

An Approximate Solution of the Jaynes–Cummings Model with Dissipation

Kazuyuki FUJII ^{*} and Tatsuo SUZUKI [†]

^{*}Department of Mathematical Sciences

Yokohama City University

Yokohama, 236–0027

Japan

[†]Center for Educational Assistance

Shibaura Institute of Technology

Saitama, 337–8570

Japan

Abstract

In this paper we treat the Jaynes–Cummings model with dissipation and give an approximate solution to the master equation for the density operator **under the general setting** by making use of the Zassenhaus expansion.

In this paper we treat the Jaynes–Cummings model ([1]) with dissipation (or the quantum damped Jaynes–Cummings model in our terminology) and study the structure of general solution from a mathematical point of view in order to apply it to Quantum Computation

^{*}E-mail address : fujii@yokohama-cu.ac.jp

[†]E-mail address : suzukita@aoni.waseda.jp ; i027110@sic.shibaura-it.ac.jp

or Quantum Control, which are our final target. As a general introduction to these topics see for example [2] and [3].

We believe that the model will become a good starting point to study more sophisticated models (with dissipation) in a near future, see for example **Concluding Remarks** in the paper.

Let us start with the following phenomenological master equation for the density operator of the atom-cavity system in [4] :

$$\frac{\partial}{\partial t}\rho = -i[H_{JC}, \rho] + \mu \left\{ a\rho a^\dagger - \frac{1}{2}(a^\dagger a\rho + \rho a^\dagger a) \right\} + \nu \left\{ a^\dagger \rho a - \frac{1}{2}(aa^\dagger \rho + \rho aa^\dagger) \right\} \quad (1)$$

where H_{JC} is the well-known Jaynes-Cummings Hamiltonian (see [1]) given by

$$\begin{aligned} H_{JC} &= \frac{\omega_0}{2}\sigma_3 \otimes \mathbf{1} + \omega_0 1_2 \otimes a^\dagger a + \Omega (\sigma_+ \otimes a + \sigma_- \otimes a^\dagger) \\ &= \begin{pmatrix} \frac{\omega_0}{2} + \omega_0 N & \Omega a \\ \Omega a^\dagger & -\frac{\omega_0}{2} + \omega_0 N \end{pmatrix} \end{aligned} \quad (2)$$

with

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

, and a and a^\dagger are the annihilation and creation operators of an electro-magnetic field mode in a cavity, $N \equiv a^\dagger a$ is the number operator, and μ and ν ($\mu > \nu \geq 0$) are some constants depending on it (for example, a damping rate of the cavity mode).

Note that the density operator ρ is in $M(2; \mathbf{C}) \otimes M(\mathcal{F}) = M(2; M(\mathcal{F}))$, namely

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \in M(2; M(\mathcal{F})) \quad (3)$$

where $M(\mathcal{F})$ is the set of all operators on the Fock space \mathcal{F} defined by

$$\begin{aligned} \mathcal{F} &\equiv \text{Vect}_{\mathbf{C}}\{|0\rangle, |1\rangle, |2\rangle, |3\rangle, \dots\} \\ &= \left\{ \sum_{n=0}^{\infty} c_n |n\rangle \mid \sum_{n=0}^{\infty} |c_n|^2 < \infty \right\}; \quad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle \end{aligned}$$

and $\mathbf{1}$ is the identity operator.

We would like to solve (1) explicitly. However, it is very hard to solve at the present time, so we must satisfy by giving some approximate solution to it.

First of all we write down the equation (1) in a component-wise manner. For that we set

$$H_{JC} = \begin{pmatrix} \frac{\omega_0}{2} + \omega_0 N & \Omega a \\ \Omega a^\dagger & -\frac{\omega_0}{2} + \omega_0 N \end{pmatrix} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (4)$$

for simplicity.

Then it is easy to see

$$\begin{aligned} \dot{\rho}_{00} &= -i(A\rho_{00} + B\rho_{10} - \rho_{00}A - \rho_{01}C) + \\ &\quad \mu \left\{ a\rho_{00}a^\dagger - \frac{1}{2}(a^\dagger a\rho_{00} + \rho_{00}a^\dagger a) \right\} + \nu \left\{ a^\dagger \rho_{00}a - \frac{1}{2}(aa^\dagger \rho_{00} + \rho_{00}aa^\dagger) \right\}, \\ \dot{\rho}_{01} &= -i(A\rho_{01} + B\rho_{11} - \rho_{00}B - \rho_{01}D) + \\ &\quad \mu \left\{ a\rho_{01}a^\dagger - \frac{1}{2}(a^\dagger a\rho_{01} + \rho_{01}a^\dagger a) \right\} + \nu \left\{ a^\dagger \rho_{01}a - \frac{1}{2}(aa^\dagger \rho_{01} + \rho_{01}aa^\dagger) \right\}, \\ \dot{\rho}_{10} &= -i(C\rho_{00} + D\rho_{10} - \rho_{10}A - \rho_{11}C) + \\ &\quad \mu \left\{ a\rho_{10}a^\dagger - \frac{1}{2}(a^\dagger a\rho_{10} + \rho_{10}a^\dagger a) \right\} + \nu \left\{ a^\dagger \rho_{10}a - \frac{1}{2}(aa^\dagger \rho_{10} + \rho_{10}aa^\dagger) \right\}, \\ \dot{\rho}_{11} &= -i(C\rho_{01} + D\rho_{11} - \rho_{10}B - \rho_{11}D) + \\ &\quad \mu \left\{ a\rho_{11}a^\dagger - \frac{1}{2}(a^\dagger a\rho_{11} + \rho_{11}a^\dagger a) \right\} + \nu \left\{ a^\dagger \rho_{11}a - \frac{1}{2}(aa^\dagger \rho_{11} + \rho_{11}aa^\dagger) \right\} \end{aligned} \quad (5)$$

where $\dot{\rho}_{ij} = (\partial/\partial t)\rho_{ij}$ as usual.

Here let us remind some technique used in [5] and [6], which is very useful in some case.

For a matrix $X = (x_{ij}) \in M(\mathcal{F})$

$$X = \begin{pmatrix} x_{00} & x_{01} & x_{02} & \cdots \\ x_{10} & x_{11} & x_{12} & \cdots \\ x_{20} & x_{21} & x_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

we correspond to the vector $\hat{X} \in \mathcal{F}^{\dim_{\mathcal{C}} \mathcal{F}}$ as

$$X = (x_{ij}) \longrightarrow \hat{X} = (x_{00}, x_{01}, x_{02}, \cdots; x_{10}, x_{11}, x_{12}, \cdots; x_{20}, x_{21}, x_{22}, \cdots; \cdots)^T \quad (6)$$

where T means the transpose. Then the following formula

$$\widehat{EXF} = (E \otimes F^T) \widehat{X} \quad (7)$$

holds for $E, F, X \in M(\mathcal{F})$.

This and equations (5) give

$$\begin{aligned} \dot{\hat{\rho}}_{00} &= -i \left\{ (A \otimes \mathbf{1} - \mathbf{1} \otimes A^T) \hat{\rho}_{00} - \mathbf{1} \otimes C^T \hat{\rho}_{01} + B \otimes \mathbf{1} \hat{\rho}_{10} \right\} + \\ &\quad \left[\mu \left\{ a \otimes (a^\dagger)^T - \frac{1}{2} (a^\dagger a \otimes \mathbf{1} + \mathbf{1} \otimes a^\dagger a) \right\} + \nu \left\{ a^\dagger \otimes a^T - \frac{1}{2} (aa^\dagger \otimes \mathbf{1} + \mathbf{1} \otimes aa^\dagger) \right\} \right] \hat{\rho}_{00}, \\ \dot{\hat{\rho}}_{01} &= -i \left\{ -\mathbf{1} \otimes B^T \hat{\rho}_{00} + (A \otimes \mathbf{1} - \mathbf{1} \otimes D^T) \hat{\rho}_{01} + B \otimes \mathbf{1} \hat{\rho}_{11} \right\} + \\ &\quad \left[\mu \left\{ a \otimes (a^\dagger)^T - \frac{1}{2} (a^\dagger a \otimes \mathbf{1} + \mathbf{1} \otimes a^\dagger a) \right\} + \nu \left\{ a^\dagger \otimes a^T - \frac{1}{2} (aa^\dagger \otimes \mathbf{1} + \mathbf{1} \otimes aa^\dagger) \right\} \right] \hat{\rho}_{01}, \\ \dot{\hat{\rho}}_{10} &= -i \left\{ C \otimes \mathbf{1} \hat{\rho}_{00} + (D \otimes \mathbf{1} - \mathbf{1} \otimes A^T) \hat{\rho}_{10} - \mathbf{1} \otimes C^T \hat{\rho}_{11} \right\} + \\ &\quad \left[\mu \left\{ a \otimes (a^\dagger)^T - \frac{1}{2} (a^\dagger a \otimes \mathbf{1} + \mathbf{1} \otimes a^\dagger a) \right\} + \nu \left\{ a^\dagger \otimes a^T - \frac{1}{2} (aa^\dagger \otimes \mathbf{1} + \mathbf{1} \otimes aa^\dagger) \right\} \right] \hat{\rho}_{10}, \\ \dot{\hat{\rho}}_{11} &= -i \left\{ C \otimes \mathbf{1} \hat{\rho}_{01} - \mathbf{1} \otimes B^T \hat{\rho}_{10} + (D \otimes \mathbf{1} - \mathbf{1} \otimes D^T) \hat{\rho}_{11} \right\} + \\ &\quad \left[\mu \left\{ a \otimes (a^\dagger)^T - \frac{1}{2} (a^\dagger a \otimes \mathbf{1} + \mathbf{1} \otimes a^\dagger a) \right\} + \nu \left\{ a^\dagger \otimes a^T - \frac{1}{2} (aa^\dagger \otimes \mathbf{1} + \mathbf{1} \otimes aa^\dagger) \right\} \right] \hat{\rho}_{11} \end{aligned} \quad (8)$$

because $\mathbf{1}$ and $N = a^\dagger a$ are diagonal ($\mathbf{1}^T = \mathbf{1}$, $N^T = N$).

From (3) we set

$$\hat{\rho} = \begin{pmatrix} \hat{\rho}_{00} \\ \hat{\rho}_{01} \\ \hat{\rho}_{10} \\ \hat{\rho}_{11} \end{pmatrix} \Longleftarrow \rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}, \quad (9)$$

then we obtain the following “canonical” form

$$\begin{aligned}
\frac{\partial}{\partial t}\hat{\rho} = & -i \begin{pmatrix} A \otimes \mathbf{1} - \mathbf{1} \otimes A^T & -\mathbf{1} \otimes C^T & B \otimes \mathbf{1} & 0 \\ -\mathbf{1} \otimes B^T & A \otimes \mathbf{1} - \mathbf{1} \otimes D^T & 0 & B \otimes \mathbf{1} \\ C \otimes \mathbf{1} & 0 & D \otimes \mathbf{1} - \mathbf{1} \otimes A^T & -\mathbf{1} \otimes C^T \\ 0 & C \otimes \mathbf{1} & -\mathbf{1} \otimes B^T & D \otimes \mathbf{1} - \mathbf{1} \otimes D^T \end{pmatrix} \hat{\rho} \\
& + \begin{pmatrix} L & & & \\ & L & & \\ & & L & \\ & & & L \end{pmatrix} \hat{\rho}
\end{aligned} \tag{10}$$

with

$$L = \mu \left\{ a \otimes (a^\dagger)^T - \frac{1}{2}(a^\dagger a \otimes \mathbf{1} + \mathbf{1} \otimes a^\dagger a) \right\} + \nu \left\{ a^\dagger \otimes a^T - \frac{1}{2}(aa^\dagger \otimes \mathbf{1} + \mathbf{1} \otimes aa^\dagger) \right\},$$

or more explicitly

$$\begin{aligned}
\frac{\partial}{\partial t}\hat{\rho} = & -i \begin{pmatrix} \omega_0(N \otimes \mathbf{1} - \mathbf{1} \otimes N) & -\Omega \mathbf{1} \otimes (a^\dagger)^T & \Omega a \otimes \mathbf{1} & 0 \\ -\Omega \mathbf{1} \otimes a^T & \omega_0 + \omega_0(N \otimes \mathbf{1} - \mathbf{1} \otimes N) & 0 & \Omega a \otimes \mathbf{1} \\ \Omega a^\dagger \otimes \mathbf{1} & 0 & -\omega_0 + \omega_0(N \otimes \mathbf{1} - \mathbf{1} \otimes N) & -\Omega \mathbf{1} \otimes (a^\dagger)^T \\ 0 & \Omega a^\dagger \otimes \mathbf{1} & -\Omega \mathbf{1} \otimes a^T & \omega_0(N \otimes \mathbf{1} - \mathbf{1} \otimes N) \end{pmatrix} \hat{\rho} \\
& + \begin{pmatrix} L & & & \\ & L & & \\ & & L & \\ & & & L \end{pmatrix} \hat{\rho}
\end{aligned} \tag{11}$$

from (4) :

$$A = \frac{\omega_0}{2} + \omega_0 N, \quad B = \Omega a, \quad C = \Omega a^\dagger, \quad D = -\frac{\omega_0}{2} + \omega_0 N.$$

Here a simplified notation ω_0 in place of $\omega_0 \mathbf{1} \otimes \mathbf{1}$ has been used.

Now we set

$$X = \begin{pmatrix} \omega_0(N \otimes \mathbf{1} - \mathbf{1} \otimes N) & -\Omega \mathbf{1} \otimes (a^\dagger)^T & \Omega a \otimes \mathbf{1} & 0 \\ -\Omega \mathbf{1} \otimes a^T & \omega_0 + \omega_0(N \otimes \mathbf{1} - \mathbf{1} \otimes N) & 0 & \Omega a \otimes \mathbf{1} \\ \Omega a^\dagger \otimes \mathbf{1} & 0 & -\omega_0 + \omega_0(N \otimes \mathbf{1} - \mathbf{1} \otimes N) & -\Omega \mathbf{1} \otimes (a^\dagger)^T \\ 0 & \Omega a^\dagger \otimes \mathbf{1} & -\Omega \mathbf{1} \otimes a^T & \omega_0(N \otimes \mathbf{1} - \mathbf{1} \otimes N) \end{pmatrix}, \quad (12)$$

$$Y = \begin{pmatrix} L & & & \\ & L & & \\ & & L & \\ & & & L \end{pmatrix} \left(L = \mu \left\{ a \otimes (a^\dagger)^T - \frac{1}{2}(a^\dagger a \otimes \mathbf{1} + \mathbf{1} \otimes a^\dagger a) \right\} + \nu \left\{ a^\dagger \otimes a^T - \frac{1}{2}(a a^\dagger \otimes \mathbf{1} + \mathbf{1} \otimes a a^\dagger) \right\} \right) \quad (13)$$

for simplicity.

Next, in order to rewrite X and Y in terms of Lie algebraic notations used in [5] we set

$$\begin{aligned} K_+ &= a^\dagger \otimes a^T, \quad K_- = a \otimes (a^\dagger)^T, \quad K_3 = \frac{1}{2}(N \otimes \mathbf{1} + \mathbf{1} \otimes N + \mathbf{1} \otimes \mathbf{1}), \\ K_0 &= N \otimes \mathbf{1} - \mathbf{1} \otimes N. \end{aligned} \quad (14)$$

Then it is easy to see

$$\begin{aligned} (K_+)^{\dagger} &= K_-, \quad (K_3)^{\dagger} = K_3, \quad (K_0)^{\dagger} = K_0, \\ [K_3, K_+] &= K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3, \\ [K_0, K_+] &= [K_0, K_-] = [K_0, K_3] = 0 \end{aligned} \quad (15)$$

and

$$\begin{aligned} [K_0, a \otimes \mathbf{1}] + a \otimes \mathbf{1} &= 0, \quad [K_0, \mathbf{1} \otimes a^T] + \mathbf{1} \otimes a^T = 0, \\ [K_0, a^\dagger \otimes \mathbf{1}] - a^\dagger \otimes \mathbf{1} &= 0, \quad [K_0, \mathbf{1} \otimes (a^\dagger)^T] - \mathbf{1} \otimes (a^\dagger)^T = 0. \end{aligned} \quad (16)$$

Namely, $\{K_+, K_-, K_3\}$ are generators of the Lie algebra $su(1,1)$, see for example [7] as a general introduction.

Under the above notations X and Y in (12) and (13) can be rewritten as

$$X = \begin{pmatrix} \omega_0 K_0 & -\Omega \mathbf{1} \otimes (a^\dagger)^T & \Omega a \otimes \mathbf{1} & 0 \\ -\Omega \mathbf{1} \otimes a^T & \omega_0 + \omega_0 K_0 & 0 & \Omega a \otimes \mathbf{1} \\ \Omega a^\dagger \otimes \mathbf{1} & 0 & -\omega_0 + \omega_0 K_0 & -\Omega \mathbf{1} \otimes (a^\dagger)^T \\ 0 & \Omega a^\dagger \otimes \mathbf{1} & -\Omega \mathbf{1} \otimes a^T & \omega_0 K_0 \end{pmatrix} \quad (17)$$

and

$$Y = \begin{pmatrix} L & & & \\ & L & & \\ & & L & \\ & & & L \end{pmatrix} \left(L = \nu K_+ + \mu K_- - (\mu + \nu) K_3 + \frac{\mu - \nu}{2} \right) \\ \equiv \frac{\mu - \nu}{2} + (\nu K_+ + \mu K_- - (\mu + \nu) K_3) 1_4 \quad (\text{for simplicity}) \quad (18)$$

where in the process the relation $aa^\dagger = a^\dagger a + \mathbf{1} = N + \mathbf{1}$ has been used.

As a result we obtain the final form

$$\frac{\partial}{\partial t} \hat{\rho} = (-iX + Y) \hat{\rho} \quad (19)$$

with X in (17) and Y in (18). This is our main result and we believe that the equation is clear-cut enough (compare it with (1)).

Since the general solution is given by

$$\hat{\rho}(t) = e^{t(-iX+Y)} \hat{\rho}(0) \quad (20)$$

in a formal way we must calculate the term $e^{t(-iX+Y)}$, which is in general very hard. For that the following Zassenhaus formula is convenient.

Zassenhaus Formula We have an expansion

$$e^{t(A+B)} = \dots e^{-\frac{t^3}{6}\{2[[A,B],B]+[[A,B],A]\}} e^{\frac{t^2}{2}[A,B]} e^{tB} e^{tA}. \quad (21)$$

The formula is a bit different from that of [8].

In this paper we use the approximation

$$e^{t(A+B)} \approx e^{\frac{t^2}{2}[A,B]} e^{tB} e^{tA}$$

with $A = -iX$ and $B = Y$, so that $\hat{\rho}(t)$ is approximated as

$$\hat{\rho}(t) \approx e^{-i\frac{t^2}{2}[X,Y]} e^{tY} e^{-itX} \hat{\rho}(0). \quad (22)$$

Let us calculate each term explicitly :

[I] First, we calculate e^{-itX} . However, the calculation is more or less well-known. We decompose X into two parts

$$\begin{aligned} X_1 &= \begin{pmatrix} \omega_0 K_0 & -\Omega \mathbf{1} \otimes (a^\dagger)^T & 0 & 0 \\ -\Omega \mathbf{1} \otimes a^T & \omega_0 + \omega_0 K_0 & 0 & 0 \\ 0 & 0 & -\omega_0 + \omega_0 K_0 & -\Omega \mathbf{1} \otimes (a^\dagger)^T \\ 0 & 0 & -\Omega \mathbf{1} \otimes a^T & \omega_0 K_0 \end{pmatrix}, \\ X_2 &= \begin{pmatrix} 0 & 0 & \Omega a \otimes \mathbf{1} & 0 \\ 0 & 0 & 0 & \Omega a \otimes \mathbf{1} \\ \Omega a^\dagger \otimes \mathbf{1} & 0 & 0 & 0 \\ 0 & \Omega a^\dagger \otimes \mathbf{1} & 0 & 0 \end{pmatrix}. \end{aligned} \quad (23)$$

Then it is not difficult to check

$$[X_1, X_2] = 0 \quad (24)$$

owing to the relations (16). If we set

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (25)$$

then

$$SX_2S^{-1} = SX_2S = \Omega \begin{pmatrix} 0 & a \otimes \mathbf{1} & 0 & 0 \\ a^\dagger \otimes \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & a \otimes \mathbf{1} \\ 0 & 0 & a^\dagger \otimes \mathbf{1} & 0 \end{pmatrix}. \quad (26)$$

Moreover, if we decompose like

$$\begin{pmatrix} \omega_0 K_0 & -\Omega \mathbf{1} \otimes (a^\dagger)^T \\ -\Omega \mathbf{1} \otimes a^T & \omega_0 + \omega_0 K_0 \end{pmatrix} = \begin{pmatrix} \omega_0 K_0 & 0 \\ 0 & \omega_0 + \omega_0 K_0 \end{pmatrix} + \begin{pmatrix} 0 & -\Omega \mathbf{1} \otimes (a^\dagger)^T \\ -\Omega \mathbf{1} \otimes a^T & 0 \end{pmatrix}$$

then

$$\left[\begin{pmatrix} \omega_0 K_0 & 0 \\ 0 & \omega_0 + \omega_0 K_0 \end{pmatrix}, \begin{pmatrix} 0 & -\Omega \mathbf{1} \otimes (a^\dagger)^T \\ -\Omega \mathbf{1} \otimes a^T & 0 \end{pmatrix} \right] = 0$$

owing to the relations (16). Therefore we have a decomposition

$$X_1 = \begin{pmatrix} \omega_0 K_0 & 0 & 0 & 0 \\ 0 & \omega_0 + \omega_0 K_0 & 0 & 0 \\ 0 & 0 & -\omega_0 + \omega_0 K_0 & 0 \\ 0 & 0 & 0 & \omega_0 K_0 \end{pmatrix} - \Omega \begin{pmatrix} 0 & \mathbf{1} \otimes (a^\dagger)^T & 0 & 0 \\ \mathbf{1} \otimes a^T & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \otimes (a^\dagger)^T \\ 0 & 0 & \mathbf{1} \otimes a^T & 0 \end{pmatrix}. \quad (27)$$

As a result we have a decomposition consisting of three commutative operators

$$X = \begin{pmatrix} \omega_0 K_0 & 0 & 0 & 0 \\ 0 & \omega_0 + \omega_0 K_0 & 0 & 0 \\ 0 & 0 & -\omega_0 + \omega_0 K_0 & 0 \\ 0 & 0 & 0 & \omega_0 K_0 \end{pmatrix} - \Omega \begin{pmatrix} 0 & \mathbf{1} \otimes (a^\dagger)^T & 0 & 0 \\ \mathbf{1} \otimes a^T & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \otimes (a^\dagger)^T \\ 0 & 0 & \mathbf{1} \otimes a^T & 0 \end{pmatrix} + \Omega \begin{pmatrix} 0 & a \otimes \mathbf{1} & 0 & 0 \\ a^\dagger \otimes \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & a \otimes \mathbf{1} \\ 0 & 0 & a^\dagger \otimes \mathbf{1} & 0 \end{pmatrix} S. \quad (28)$$

Now, by making use of the well-known formula

$$\exp \left(-i\alpha \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix} \right) = \begin{pmatrix} \cos(\alpha\sqrt{AA^\dagger}) & -i\frac{1}{\sqrt{AA^\dagger}} \sin(\alpha\sqrt{AA^\dagger})A \\ -i\frac{1}{\sqrt{A^\dagger A}} \sin(\alpha\sqrt{A^\dagger A})A^\dagger & \cos(\alpha\sqrt{A^\dagger A}) \end{pmatrix} \quad (29)$$

where α is some parameter, we can calculate e^{-itX} easily. The result is

$$e^{-itX} = \begin{pmatrix} (11) & (12) & (13) & (14) \\ (21) & (22) & (23) & (24) \\ (31) & (32) & (33) & (34) \\ (41) & (42) & (43) & (44) \end{pmatrix} \quad (30)$$

where for $\alpha = \Omega t$

$$\begin{aligned} (11) &= e^{-it\omega_0 K_0} \cos(\alpha\sqrt{\mathbf{1} \otimes aa^\dagger}) \cos(\alpha\sqrt{aa^\dagger \otimes \mathbf{1}}) \\ &= e^{-it\omega_0 N} \cos(\alpha\sqrt{N+1}) \otimes e^{it\omega_0 N} \cos(\alpha\sqrt{N+1}), \\ (12) &= i e^{-it\omega_0 K_0} \frac{1}{\sqrt{\mathbf{1} \otimes aa^\dagger}} \sin(\alpha\sqrt{\mathbf{1} \otimes aa^\dagger})(\mathbf{1} \otimes (a^\dagger)^T) \cos(\alpha\sqrt{aa^\dagger \otimes \mathbf{1}}) \\ &= i e^{-it\omega_0 N} \cos(\alpha\sqrt{N+1}) \otimes e^{it\omega_0 N} \frac{1}{\sqrt{N+1}} \sin(\alpha\sqrt{N+1})(a^\dagger)^T, \\ (13) &= -i e^{-it\omega_0 K_0} \cos(\alpha\sqrt{\mathbf{1} \otimes aa^\dagger}) \frac{1}{\sqrt{aa^\dagger \otimes \mathbf{1}}} \sin(\alpha\sqrt{aa^\dagger \otimes \mathbf{1}})(a \otimes \mathbf{1}) \\ &= -i e^{-it\omega_0 N} \frac{1}{\sqrt{N+1}} \sin(\alpha\sqrt{N+1})a \otimes e^{it\omega_0 N} \cos(\alpha\sqrt{N+1}), \\ (14) &= e^{-it\omega_0 K_0} \frac{1}{\sqrt{\mathbf{1} \otimes aa^\dagger}} \sin(\alpha\sqrt{\mathbf{1} \otimes aa^\dagger})(\mathbf{1} \otimes (a^\dagger)^T) \frac{1}{\sqrt{aa^\dagger \otimes \mathbf{1}}} \sin(\alpha\sqrt{aa^\dagger \otimes \mathbf{1}})(a \otimes \mathbf{1}) \\ &= e^{-it\omega_0 N} \frac{1}{\sqrt{N+1}} \sin(\alpha\sqrt{N+1})a \otimes e^{it\omega_0 N} \frac{1}{\sqrt{N+1}} \sin(\alpha\sqrt{N+1})(a^\dagger)^T; \end{aligned}$$

and

$$\begin{aligned}
(21) &= ie^{-it\omega_0} e^{-it\omega_0 K_0} \frac{1}{\sqrt{\mathbf{1} \otimes a^\dagger a}} \sin(\alpha \sqrt{\mathbf{1} \otimes a^\dagger a}) (\mathbf{1} \otimes a^T) \cos(\alpha \sqrt{aa^\dagger \otimes \mathbf{1}}) \\
&= ie^{-it\omega_0} e^{-it\omega_0 N} \cos(\alpha \sqrt{N+1}) \otimes e^{it\omega_0 N} \frac{1}{\sqrt{N}} \sin(\alpha \sqrt{N}) a^T, \\
(22) &= e^{-it\omega_0} e^{-it\omega_0 K_0} \cos(\alpha \sqrt{\mathbf{1} \otimes a^\dagger a}) \cos(\alpha \sqrt{aa^\dagger \otimes \mathbf{1}}) \\
&= e^{-it\omega_0} e^{-it\omega_0 N} \cos(\alpha \sqrt{N+1}) \otimes e^{it\omega_0 N} \cos(\alpha \sqrt{N}), \\
(23) &= e^{-it\omega_0} e^{-it\omega_0 K_0} \frac{1}{\sqrt{\mathbf{1} \otimes a^\dagger a}} \sin(\alpha \sqrt{\mathbf{1} \otimes a^\dagger a}) (\mathbf{1} \otimes a^T) \frac{1}{\sqrt{aa^\dagger \otimes \mathbf{1}}} \sin(\alpha \sqrt{aa^\dagger \otimes \mathbf{1}}) (a \otimes \mathbf{1}) \\
&= e^{-it\omega_0} e^{-it\omega_0 N} \frac{1}{\sqrt{N+1}} \sin(\alpha \sqrt{N+1}) a \otimes e^{it\omega_0 N} \frac{1}{\sqrt{N}} \sin(\alpha \sqrt{N}) a^T, \\
(24) &= -ie^{-it\omega_0} e^{-it\omega_0 K_0} \cos(\alpha \sqrt{\mathbf{1} \otimes a^\dagger a}) \frac{1}{\sqrt{aa^\dagger \otimes \mathbf{1}}} \sin(\alpha \sqrt{aa^\dagger \otimes \mathbf{1}}) (a \otimes \mathbf{1}) \\
&= -ie^{-it\omega_0} e^{-it\omega_0 N} \frac{1}{\sqrt{N+1}} \sin(\alpha \sqrt{N+1}) a \otimes e^{it\omega_0 N} \cos(\alpha \sqrt{N});
\end{aligned}$$

and

$$\begin{aligned}
(31) &= -ie^{it\omega_0} e^{-it\omega_0 K_0} \cos(\alpha \sqrt{\mathbf{1} \otimes aa^\dagger}) \frac{1}{\sqrt{a^\dagger a \otimes \mathbf{1}}} \sin(\alpha \sqrt{a^\dagger a \otimes \mathbf{1}}) (a^\dagger \otimes \mathbf{1}) \\
&= -ie^{it\omega_0} e^{-it\omega_0 N} \frac{1}{\sqrt{N}} \sin(\alpha \sqrt{N}) a^\dagger \otimes e^{it\omega_0 N} \cos(\alpha \sqrt{N+1}), \\
(32) &= e^{it\omega_0} e^{-it\omega_0 K_0} \frac{1}{\sqrt{\mathbf{1} \otimes aa^\dagger}} \sin(\alpha \sqrt{\mathbf{1} \otimes aa^\dagger}) (\mathbf{1} \otimes (a^\dagger)^T) \frac{1}{\sqrt{a^\dagger a \otimes \mathbf{1}}} \sin(\alpha \sqrt{a^\dagger a \otimes \mathbf{1}}) (a^\dagger \otimes \mathbf{1}) \\
&= e^{it\omega_0} e^{-it\omega_0 N} \frac{1}{\sqrt{N}} \sin(\alpha \sqrt{N}) a^\dagger \otimes e^{it\omega_0 N} \frac{1}{\sqrt{N+1}} \sin(\alpha \sqrt{N+1}) (a^\dagger)^T, \\
(33) &= e^{it\omega_0} e^{-it\omega_0 K_0} \cos(\alpha \sqrt{\mathbf{1} \otimes aa^\dagger}) \cos(\alpha \sqrt{a^\dagger a \otimes \mathbf{1}}) \\
&= e^{it\omega_0} e^{-it\omega_0 N} \cos(\alpha \sqrt{N}) \otimes e^{it\omega_0 N} \cos(\alpha \sqrt{N+1}), \\
(34) &= ie^{it\omega_0} e^{-it\omega_0 K_0} \frac{1}{\sqrt{\mathbf{1} \otimes aa^\dagger}} \sin(\alpha \sqrt{\mathbf{1} \otimes aa^\dagger}) (\mathbf{1} \otimes (a^\dagger)^T) \cos(\alpha \sqrt{a^\dagger a \otimes \mathbf{1}}) \\
&= ie^{it\omega_0} e^{-it\omega_0 N} \cos(\alpha \sqrt{N}) \otimes e^{it\omega_0 N} \frac{1}{\sqrt{N+1}} \sin(\alpha \sqrt{N+1}) (a^\dagger)^T;
\end{aligned}$$

and

$$\begin{aligned}
(41) &= e^{-it\omega_0 K_0} \frac{1}{\sqrt{\mathbf{1} \otimes a^\dagger a}} \sin(\alpha \sqrt{\mathbf{1} \otimes a^\dagger a}) (\mathbf{1} \otimes a^T) \frac{1}{\sqrt{a^\dagger a \otimes \mathbf{1}}} \sin(\alpha \sqrt{a^\dagger a \otimes \mathbf{1}}) (a^\dagger \otimes \mathbf{1}) \\
&= e^{-it\omega_0 N} \frac{1}{\sqrt{N}} \sin(\alpha \sqrt{N}) a^\dagger \otimes e^{it\omega_0 N} \frac{1}{\sqrt{N}} \sin(\alpha \sqrt{N}) a^T, \\
(42) &= -ie^{-it\omega_0 K_0} \cos(\alpha \sqrt{\mathbf{1} \otimes a^\dagger a}) \frac{1}{\sqrt{a^\dagger a \otimes \mathbf{1}}} \sin(\alpha \sqrt{a^\dagger a \otimes \mathbf{1}}) (a^\dagger \otimes \mathbf{1}) \\
&= -i e^{-it\omega_0 N} \frac{1}{\sqrt{N}} \sin(\alpha \sqrt{N}) a^\dagger \otimes e^{it\omega_0 N} \cos(\alpha \sqrt{N}), \\
(43) &= ie^{-it\omega_0 K_0} \frac{1}{\sqrt{\mathbf{1} \otimes a^\dagger a}} \sin(\alpha \sqrt{\mathbf{1} \otimes a^\dagger a}) (\mathbf{1} \otimes a^T) \cos(\alpha \sqrt{a^\dagger a \otimes \mathbf{1}}) \\
&= i e^{-it\omega_0 N} \cos(\alpha \sqrt{N}) \otimes e^{it\omega_0 N} \frac{1}{\sqrt{N}} \sin(\alpha \sqrt{N}) a^T, \\
(44) &= e^{-it\omega_0 K_0} \cos(\alpha \sqrt{\mathbf{1} \otimes a^\dagger a}) \cos(\alpha \sqrt{a^\dagger a \otimes \mathbf{1}}) \\
&= e^{-it\omega_0 N} \cos(\alpha \sqrt{N}) \otimes e^{it\omega_0 N} \cos(\alpha \sqrt{N}).
\end{aligned}$$

[II] Second, we must calculate e^{tY} . To be glad this task has been done, see [5] and [6]. Namely, from (18) we have

$$e^{tY} = \begin{pmatrix} e^{tL} & & & \\ & e^{tL} & & \\ & & e^{tL} & \\ & & & e^{tL} \end{pmatrix}; \quad L = \frac{\mu - \nu}{2} + \nu K_+ + \mu K_- - (\mu + \nu) K_3 \quad (31)$$

and

$$\begin{aligned}
e^{tL} &= e^{\frac{\mu - \nu}{2} t} e^{t\{\nu K_+ + \mu K_- - (\mu + \nu) K_3\}} \\
&= e^{\frac{\mu - \nu}{2} t} e^{G(t)K_+} e^{-2 \log(F(t))K_3} e^{E(t)K_-}
\end{aligned} \quad (32)$$

with

$$\begin{aligned}
E(t) &= \frac{\frac{2\mu}{\mu - \nu} \sinh\left(\frac{\mu - \nu}{2} t\right)}{\cosh\left(\frac{\mu - \nu}{2} t\right) + \frac{\mu + \nu}{\mu - \nu} \sinh\left(\frac{\mu - \nu}{2} t\right)}, \\
F(t) &= \cosh\left(\frac{\mu - \nu}{2} t\right) + \frac{\mu + \nu}{\mu - \nu} \sinh\left(\frac{\mu - \nu}{2} t\right), \\
G(t) &= \frac{\frac{2\nu}{\mu - \nu} \sinh\left(\frac{\mu - \nu}{2} t\right)}{\cosh\left(\frac{\mu - \nu}{2} t\right) + \frac{\mu + \nu}{\mu - \nu} \sinh\left(\frac{\mu - \nu}{2} t\right)}.
\end{aligned} \quad (33)$$

This is a kind of disentangling formula, see for example [7] as a general introduction.

[III] Third, we must calculate $e^{-i\frac{t^2}{2}[X,Y]}$. In this stage some interaction appears. Let us calculate $[X, Y]$ exactly.

Some calculation gives

$$\begin{aligned}
[\mathbf{1} \otimes a^T, \nu K_+ + \mu K_- - (\mu + \nu) K_3] &= -\mu a \otimes \mathbf{1} + \frac{\mu + \nu}{2} \mathbf{1} \otimes a^T, \\
[\mathbf{1} \otimes (a^\dagger)^T, \nu K_+ + \mu K_- - (\mu + \nu) K_3] &= \nu a^\dagger \otimes \mathbf{1} - \frac{\mu + \nu}{2} \mathbf{1} \otimes (a^\dagger)^T, \\
[a \otimes \mathbf{1}, \nu K_+ + \mu K_- - (\mu + \nu) K_3] &= \nu \mathbf{1} \otimes a^T - \frac{\mu + \nu}{2} a \otimes \mathbf{1}, \\
[a^\dagger \otimes \mathbf{1}, \nu K_+ + \mu K_- - (\mu + \nu) K_3] &= -\mu \mathbf{1} \otimes (a^\dagger)^T + \frac{\mu + \nu}{2} a^\dagger \otimes \mathbf{1}, \\
[K_0, \nu K_+ + \mu K_- - (\mu + \nu) K_3] &= 0
\end{aligned} \tag{34}$$

and for simplicity we set

$$\begin{aligned}
A &= -\nu a^\dagger \otimes \mathbf{1} + \frac{\mu + \nu}{2} \mathbf{1} \otimes (a^\dagger)^T, \\
B &= \mu a \otimes \mathbf{1} - \frac{\mu + \nu}{2} \mathbf{1} \otimes a^T, \\
C &= \nu \mathbf{1} \otimes a^T - \frac{\mu + \nu}{2} a \otimes \mathbf{1}, \\
D &= -\mu \mathbf{1} \otimes (a^\dagger)^T + \frac{\mu + \nu}{2} a^\dagger \otimes \mathbf{1}.
\end{aligned} \tag{35}$$

Note that $B \neq -A^\dagger$ and $D \neq -C^\dagger$ because of $\mu > \nu$. It is easy to see

$$\begin{aligned}
[A, C] &= [A, D] = 0, \quad [B, C] = [B, D] = 0, \\
[A, B] &= [C, D] = -\left(\frac{\mu - \nu}{2}\right)^2.
\end{aligned} \tag{36}$$

From the decomposition (28) we have a decomposition consisting of two commutative operators

$$[X, Y] = \Omega \begin{pmatrix} 0 & A & 0 & 0 \\ B & 0 & 0 & 0 \\ 0 & 0 & 0 & A \\ 0 & 0 & B & 0 \end{pmatrix} + \Omega S \begin{pmatrix} 0 & C & 0 & 0 \\ D & 0 & 0 & 0 \\ 0 & 0 & 0 & C \\ 0 & 0 & D & 0 \end{pmatrix} S \tag{37}$$

with S in (25).

Now, by making use of the formula

$$\exp \left(-i\beta \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix} \right) = \begin{pmatrix} \cos(\beta\sqrt{PQ}) & -i\frac{1}{\sqrt{PQ}} \sin(\beta\sqrt{PQ})P \\ -i\frac{1}{\sqrt{QP}} \sin(\beta\sqrt{QP})Q & \cos(\beta\sqrt{QP}) \end{pmatrix} \quad (38)$$

for some parameter β , we can calculate the term $e^{-i\frac{t^2}{2}[X,Y]}$ easily. The result is

$$e^{-i\frac{t^2}{2}[X,Y]} = \begin{pmatrix} (11) & (12) & (13) & (14) \\ (21) & (22) & (23) & (24) \\ (31) & (32) & (33) & (34) \\ (41) & (42) & (43) & (44) \end{pmatrix} \quad (39)$$

where for $\beta = \Omega\frac{t^2}{2}$

$$\begin{aligned} (11) &= \cos(\beta\sqrt{AB}) \cos(\beta\sqrt{CD}), \\ (12) &= -i\frac{1}{\sqrt{AB}} \sin(\beta\sqrt{AB})A \cos(\beta\sqrt{CD}), \\ (13) &= -i \cos(\beta\sqrt{AB})\frac{1}{\sqrt{CD}} \sin(\beta\sqrt{CD})C, \\ (14) &= -\frac{1}{\sqrt{AB}} \sin(\beta\sqrt{AB})A\frac{1}{\sqrt{CD}} \sin(\beta\sqrt{CD})C ; \\ (21) &= -i\frac{1}{\sqrt{BA}} \sin(\beta\sqrt{BA})B \cos(\beta\sqrt{CD}), \\ (22) &= \cos(\beta\sqrt{BA}) \cos(\beta\sqrt{CD}), \\ (23) &= -\frac{1}{\sqrt{BA}} \sin(\beta\sqrt{BA})B\frac{1}{\sqrt{CD}} \sin(\beta\sqrt{CD})C, \\ (24) &= -i \cos(\beta\sqrt{BA})\frac{1}{\sqrt{CD}} \sin(\beta\sqrt{CD})C ; \\ (31) &= -i \cos(\beta\sqrt{AB})\frac{1}{\sqrt{DC}} \sin(\beta\sqrt{DC})D, \\ (32) &= -\frac{1}{\sqrt{AB}} \sin(\beta\sqrt{AB})A\frac{1}{\sqrt{DC}} \sin(\beta\sqrt{DC})D, \\ (33) &= \cos(\beta\sqrt{AB}) \cos(\beta\sqrt{DC}), \\ (34) &= -i\frac{1}{\sqrt{AB}} \sin(\beta\sqrt{AB})A \cos(\beta\sqrt{DC}) ; \end{aligned}$$

$$(41) = -\frac{1}{\sqrt{BA}} \sin(\beta\sqrt{BA}) B \frac{1}{\sqrt{DC}} \sin(\beta\sqrt{DC}) D,$$

$$(42) = -i \cos(\beta\sqrt{BA}) \frac{1}{\sqrt{DC}} \sin(\beta\sqrt{DC}) D,$$

$$(43) = -i \frac{1}{\sqrt{BA}} \sin(\beta\sqrt{BA}) B \cos(\beta\sqrt{DC}),$$

$$(44) = \cos(\beta\sqrt{BA}) \cos(\beta\sqrt{DC})$$

where

$$\begin{aligned} AB &= -\mu\nu a^\dagger a \otimes \mathbf{1} + \frac{\mu+\nu}{2} \nu a^\dagger \otimes a^T + \mu \frac{\mu+\nu}{2} a \otimes (a^\dagger)^T - \left(\frac{\mu+\nu}{2}\right)^2 \mathbf{1} \otimes aa^\dagger, \\ BA &= -\mu\nu aa^\dagger \otimes \mathbf{1} + \mu \frac{\mu+\nu}{2} a \otimes (a^\dagger)^T + \frac{\mu+\nu}{2} \nu a^\dagger \otimes a^T - \left(\frac{\mu+\nu}{2}\right)^2 \mathbf{1} \otimes a^\dagger a, \\ CD &= -\mu\nu \mathbf{1} \otimes a^\dagger a + \frac{\mu+\nu}{2} \nu a^\dagger \otimes a^T + \mu \frac{\mu+\nu}{2} a \otimes (a^\dagger)^T - \left(\frac{\mu+\nu}{2}\right)^2 aa^\dagger \otimes \mathbf{1}, \\ DC &= -\mu\nu \mathbf{1} \otimes aa^\dagger + \mu \frac{\mu+\nu}{2} a \otimes (a^\dagger)^T + \frac{\mu+\nu}{2} \nu a^\dagger \otimes a^T - \left(\frac{\mu+\nu}{2}\right)^2 a^\dagger a \otimes \mathbf{1}. \end{aligned}$$

These are complicated enough.

Anyway, we have completed the task although it is very complicated.

In last, we shall restore the result to original form. If we set

$$\hat{\rho}(t) = e^{tY} e^{-itX} \hat{\rho}(0) \equiv e^{tY} \hat{\rho}_1(t), \quad (40)$$

then $\tilde{\rho}_1(t)$ is from (30) given by

$$\tilde{\rho}_1(t) = \begin{pmatrix} (11) & (12) \\ (21) & (22) \end{pmatrix} \quad (41)$$

where

$$\begin{aligned}
(11) &= e^{-it\omega_0 N} \cos(\alpha\sqrt{N+1}) \rho_{00} e^{it\omega_0 N} \cos(\alpha\sqrt{N+1}) \\
&\quad + ie^{-it\omega_0 N} \cos(\alpha\sqrt{N+1}) \rho_{01} a^\dagger e^{it\omega_0 N} \frac{1}{\sqrt{N+1}} \sin(\alpha\sqrt{N+1}) \\
&\quad - ie^{-it\omega_0 N} \frac{1}{\sqrt{N+1}} \sin(\alpha\sqrt{N+1}) a \rho_{10} e^{it\omega_0 N} \cos(\alpha\sqrt{N+1}) \\
&\quad + e^{-it\omega_0 N} \frac{1}{\sqrt{N+1}} \sin(\alpha\sqrt{N+1}) a \rho_{11} a^\dagger e^{it\omega_0 N} \frac{1}{\sqrt{N+1}} \sin(\alpha\sqrt{N+1}), \\
(12) &= ie^{-it\omega_0} e^{-it\omega_0 N} \cos(\alpha\sqrt{N+1}) \rho_{00} ae^{it\omega_0 N} \frac{1}{\sqrt{N}} \sin(\alpha\sqrt{N}) \\
&\quad + e^{-it\omega_0} e^{-it\omega_0 N} \cos(\alpha\sqrt{N+1}) \rho_{01} e^{it\omega_0 N} \cos(\alpha\sqrt{N}) \\
&\quad + e^{-it\omega_0} e^{-it\omega_0 N} \frac{1}{\sqrt{N+1}} \sin(\alpha\sqrt{N+1}) a \rho_{10} ae^{it\omega_0 N} \frac{1}{\sqrt{N}} \sin(\alpha\sqrt{N}) \\
&\quad - ie^{-it\omega_0} e^{-it\omega_0 N} \frac{1}{\sqrt{N+1}} \sin(\alpha\sqrt{N+1}) a \rho_{11} e^{it\omega_0 N} \cos(\alpha\sqrt{N}), \\
(21) &= -ie^{it\omega_0} e^{-it\omega_0 N} \frac{1}{\sqrt{N}} \sin(\alpha\sqrt{N}) a^\dagger \rho_{00} e^{it\omega_0 N} \cos(\alpha\sqrt{N+1}) \\
&\quad + e^{it\omega_0} e^{-it\omega_0 N} \frac{1}{\sqrt{N}} \sin(\alpha\sqrt{N}) a^\dagger \rho_{01} a^\dagger e^{it\omega_0 N} \frac{1}{\sqrt{N+1}} \sin(\alpha\sqrt{N+1}) \\
&\quad + e^{it\omega_0} e^{-it\omega_0 N} \cos(\alpha\sqrt{N}) \rho_{10} e^{it\omega_0 N} \cos(\alpha\sqrt{N+1}) \\
&\quad + ie^{it\omega_0} e^{-it\omega_0 N} \cos(\alpha\sqrt{N}) \rho_{11} a^\dagger e^{it\omega_0 N} \frac{1}{\sqrt{N+1}} \sin(\alpha\sqrt{N+1}), \\
(22) &= e^{-it\omega_0 N} \frac{1}{\sqrt{N}} \sin(\alpha\sqrt{N}) a^\dagger \rho_{00} ae^{it\omega_0 N} \frac{1}{\sqrt{N}} \sin(\alpha\sqrt{N}) \\
&\quad - ie^{-it\omega_0 N} \frac{1}{\sqrt{N}} \sin(\alpha\sqrt{N}) a^\dagger \rho_{01} e^{it\omega_0 N} \cos(\alpha\sqrt{N}) \\
&\quad + ie^{-it\omega_0 N} \cos(\alpha\sqrt{N}) \rho_{10} ae^{it\omega_0 N} \frac{1}{\sqrt{N}} \sin(\alpha\sqrt{N}) \\
&\quad + e^{-it\omega_0 N} \cos(\alpha\sqrt{N}) \rho_{11} e^{it\omega_0 N} \cos(\alpha\sqrt{N}),
\end{aligned}$$

and $\tilde{\rho}(t)$ is by

$$\begin{aligned}
\tilde{\rho}(t) &= \frac{e^{\frac{\mu-\nu}{2}t}}{F(t)} \sum_{n=0}^{\infty} \frac{G(t)^n}{n!} (a^\dagger)^n [\exp(\{-\log(F(t))\}N) \left\{ \sum_{m=0}^{\infty} \frac{E(t)^m}{m!} a^m \tilde{\rho}_1(t) (a^\dagger)^m \right\} \times \\
&\quad \exp(\{-\log(F(t))\}N)] a^n
\end{aligned} \tag{42}$$

in terms of $E(t), F(t), G(t)$ in (33) and $N = a^\dagger a$. See [5] and [6]. This form is relatively clear because of no interaction.

Our (approximate) solution is

$$\hat{\rho}(t) = e^{-i\frac{t^2}{2}[X,Y]} e^{tY} e^{-itX} = e^{-i\frac{t^2}{2}[X,Y]} \hat{\tilde{\rho}}(t)$$

from (40). We would like to express $\rho(t)$ like (42). From (39) we can expand each term in terms of the Taylor expansion of $\cos(X)$ and $\sin(X)$. However, the form is very ugly in the Dirac's sense, so we leave such a task to readers.

Concluding Remarks In this paper we treated the master equation for the density operator based on the Jaynes–Cummings Hamiltonian with dissipation and constructed the approximate solution up to $O(t^3)$ under the general setting. Our construction is quite general because $\hat{\rho}(0)$ is any initial state.

However, it may be still inconvenient to apply to realistic problems coming from the atom–cavity system. Further work will be required.

Moreover, it may be possible to treat a generalized master equation for the density operator given by

$$\begin{aligned} \frac{\partial}{\partial t} \rho = & -i[H_{JC}, \rho] + \mu \left\{ a\rho a^\dagger - \frac{1}{2}(a^\dagger a\rho + \rho a^\dagger a) \right\} + \nu \left\{ a^\dagger \rho a - \frac{1}{2}(a a^\dagger \rho + \rho a a^\dagger) \right\} + \\ & \kappa \left\{ a\rho a - \frac{1}{2}(a^2 \rho + \rho a^2) \right\} + \bar{\kappa} \left\{ a^\dagger \rho a^\dagger - \frac{1}{2}((a^\dagger)^2 \rho + \rho (a^\dagger)^2) \right\} \end{aligned} \quad (43)$$

with the positivity condition $\mu\nu \geq |\kappa|^2$. However, we don't treat this general case in the paper, see [9], [10] and [11].

References

- [1] E. T. Jaynes and F. W. Cummings : Comparison of Quantum and Semiclassical Radiation Theories with Applications to the Beam Maser, Proc. IEEE, **51** (1963), 89.
- [2] H. -P. Breuer and F. Petruccione : The theory of open quantum systems, Oxford University Press, New York, 2002.

- [3] W. P. Schleich : Quantum Optics in Phase Space, WILEY–VCH, Berlin, 2001.
- [4] M. Scala, B. Militello, A. Messina, S. Maniscalco, J. Piilo and K.-A. Suominen : Cavity losses for the dissipative Jaynes-Cummings Hamiltonian beyond Rotating Wave Approximation, J. Phys. A: Math. Theor. **40** (2007), 14527, arXiv:0709.1614 [quant-ph].
- [5] R. Endo, K. Fujii and T. Suzuki : General Solution of the Quantum Damped Harmonic Oscillator, Int. J. Geom. Meth. Mod. Phys, **5** (2008), 653, arXiv : 0710.2724 [quant-ph].
- [6] K. Fujii and T. Suzuki : General Solution of the Quantum Damped Harmonic Oscillator II : Some Examples, Int. J. Geom. Meth. Mod. Phys, **6** (2009), 225, arXiv : 0806.2169 [quant-ph].
- [7] K. Fujii : Introduction to Coherent States and Quantum Information Theory, quant-ph/0112090.
- [8] C. Zachos : Crib Notes on Campbell-Baker-Hausdorff expansions, unpublished, 1999, see <http://www.hep.anl.gov/czachos/index.html>.
- [9] K. Fujii : Algebraic Structure of a Master Equation with Generalized Lindblad Form, Int. J. Geom. Meth. Mod. Phys, **5** (2008), 1033, arXiv : 0802.3252 [quant-ph].
- [10] K. Fujii : A Master Equation with Generalized Lindblad Form and a Unitary Transformation by the Squeezing Operator, arXiv : 0803.3105 [quant-ph].
- [11] K. Fujii : in progress.